ECE 447: Robotics Engineering

Lecture 7: Inverse Kinematics of Robot Manipulators

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Lecture Outline:

1. IK Problem Formulation.
2. Geometrical Approach.
3. Algebraic Approach.

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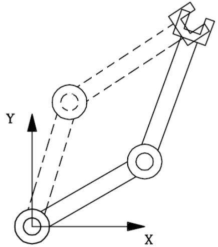
IK Problem Formulation:

Given: Desired position and orientation of end-effector, **p**.
Required: Joint Variables **q** (θ or d) to get **p**

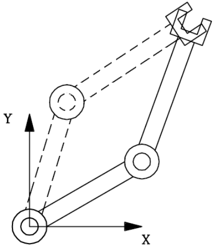
$$\mathbf{q} = f(\mathbf{p})$$

Inverse Kinematics



IK Problem Formulation:

Inverse Kinematics



Given: Desired position and orientation of end-effector, **p**.
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$$\mathbf{q} = f(\mathbf{p})$$

Given the desired 4×4 homogeneous transformation H

$$H = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix}$$

IK Problem Formulation:

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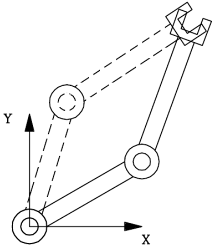

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Given the desired 4×4 homogeneous transformation H

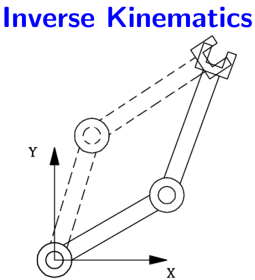
$$H = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix}$$

The task is to find a solution (possibly one of many) of the equation:

$$T_n^0(q_1, \ldots, q_n) = A_1(q_1)\ A_2(q_2) \ldots A_n(q_n) = H$$

It is 12 equations with respect to n variables $q_1, \ldots, q_n$

IK Problem Formulation:



Given: Desired position and orientation of end-effector, **p**.
Required: Joint Variables **q** (θ or d) to get **p**

$$\mathbf{q} = f(\mathbf{p})$$

It is often advantageous to find $q_1, \ldots, q_n$ in a **closed rather than numerical** form:

$$q_k = f_k(h_{11}, h_{12}, \ldots, h_{34}),\ k = 1, \ldots, n$$

This will allow **fast and deterministic computation** of the joints variables instead of searching for a possible solution.

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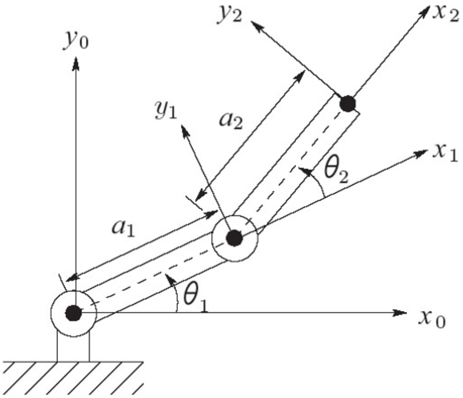
Geometrical Approach:

Example: Two link manipulator

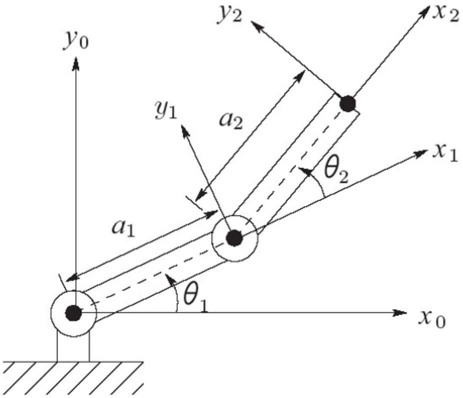
link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2



Geometrical Approach:

Example: Two link manipulator

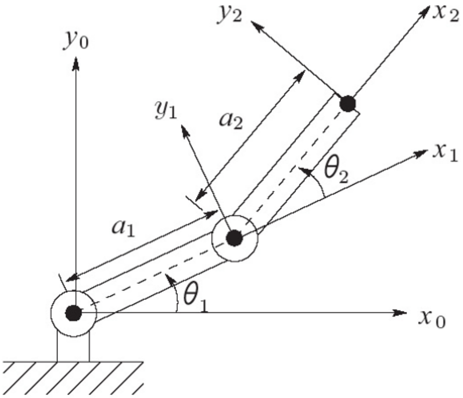
$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Geometrical Approach:

Example: Two link manipulator

$$T_2^0 = A_1 A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Forward Kinematics:

$$x = a_1 \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2)$$

$$y = a_1 \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2)$$

Geometrical Approach:

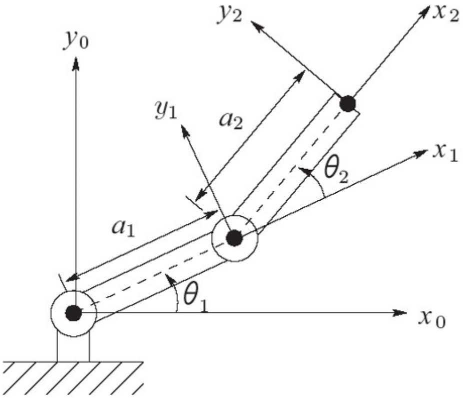
Example: Two link manipulator

$$x = a_1 \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2)$$

$$y = a_1 \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2)$$

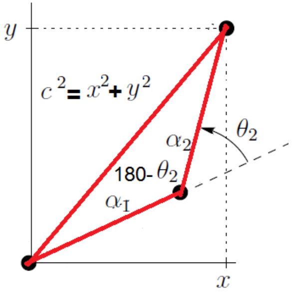
In Inverse Kinematics, we need the joint variables θ_1, θ_2 in terms of the given x and y.

- The Forward Kinematic equation are **non-linear** of sine and cosine terms.
- It is not easy to find a solution or a unique solution in general.



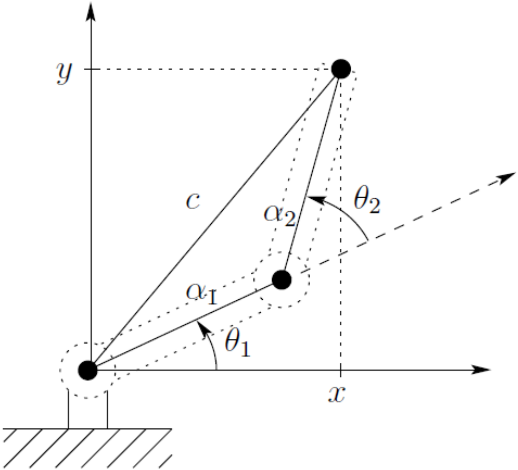
Geometrical Approach:

Example: Two link manipulator



$$c^2 = a_1^2 + a_2^2 - 2a_1a_2\cos(180 - \theta_2)$$

$$\cos(\theta_2) = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2} := D$$



Geometrical Approach:

Example: Two link manipulator

If $cos\theta_2$ is known,

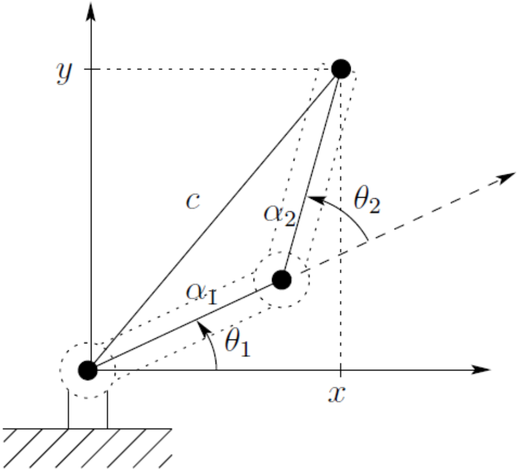
$$\sin(\theta_2) = \pm\sqrt{1 - D^2}$$

So,

$$\boxed{\theta_2 = \tan^{-1}\frac{\pm\sqrt{1 - D^2}}{D}}$$ Two Solutions

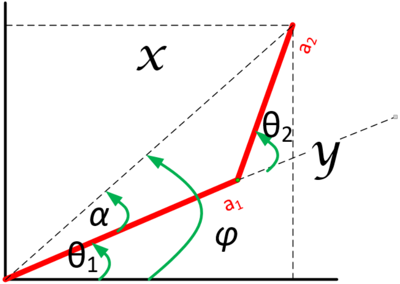
where,

$$D = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2}$$



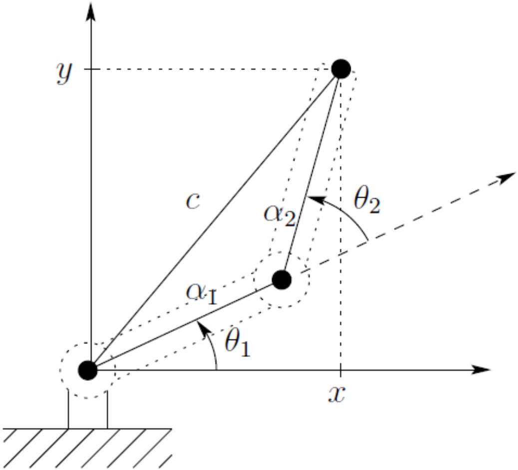
Geometrical Approach:

Example: Two link manipulator



$$\theta_1 = \phi - \alpha$$

$$\theta_1 = \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{a_2 \sin\theta_2}{a_1 + a_2\cos\theta_2}$$

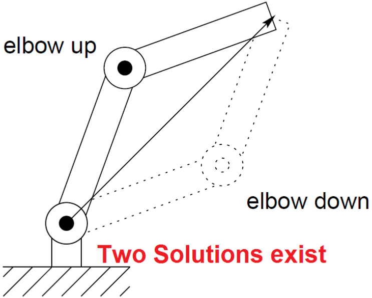


Geometrical Approach:

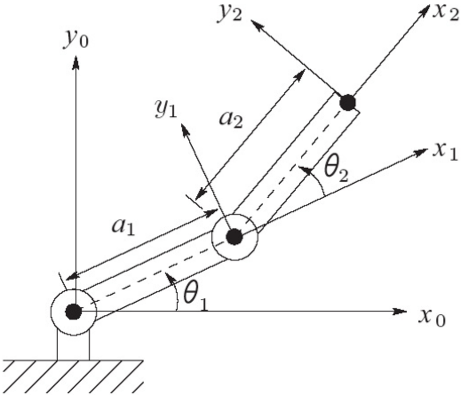
Example: Two link manipulator

$$\theta_2 = \tan^{-1}\frac{\pm\sqrt{1-D^2}}{D}; \quad D = \frac{x^2+y^2-a_1^2-a_2^2}{2a_1a_2}$$

$$\theta_1 = \tan^{-1}\frac{y}{x} - tan^{-1}\frac{a_2\sin\theta_2}{a_1+a_2\cos\theta_2}$$



Two Solutions exist



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Algebraic Approach:

Example: RPP Robot

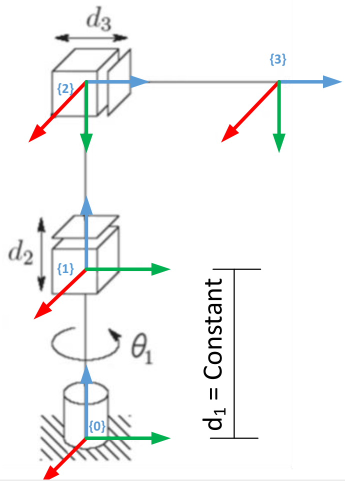
$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics:

$$x = -d_3 \sin\theta_1$$

$$y = d_3 \cos\theta_1$$

$$z = d_1 + d_2$$



Algebraic Approach:

Example: RPP Robot

$$x = -d_3 \sin \theta_1$$

$$y = d_3 \cos \theta_1$$

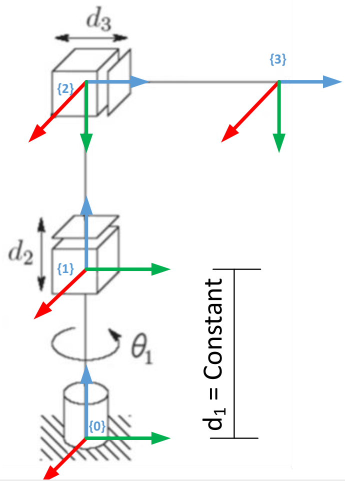
$$z = d_1 + d_2$$

Inverse Kinematics:

$$\theta_1 = \tan^{-1} \frac{-x}{y}$$

$$d_2 = z - d_1$$

$$d_3 = \sqrt{x^2 + y^2}$$



Algebraic Approach:

Example: RP Robot

$$A_2^0 = A_1^0 * A_2^1$$

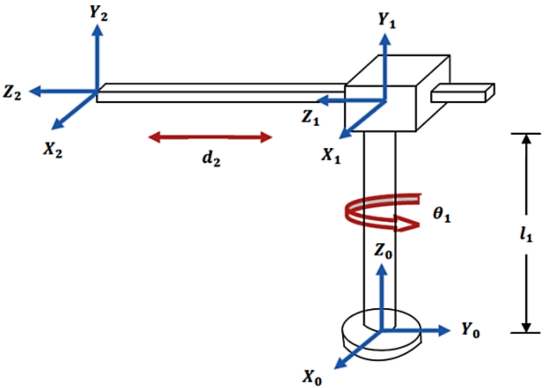
$$A_1^0 = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 & 0 \\ S\theta_1 & 0 & -C\theta_1 & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_2^1 = \begin{bmatrix} 1 & 0 & S\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2^0 = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 & d_2S\theta_1 \\ S\theta_1 & 0 & -C\theta_1 & -d_2C\theta_1 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x = d_2 \sin\theta_1$$

$$y = -d_2 \cos\theta_1$$

$$z = l_1$$



Algebraic Approach:

Example: RP Robot

$$x = d_2 \sin\theta_1$$

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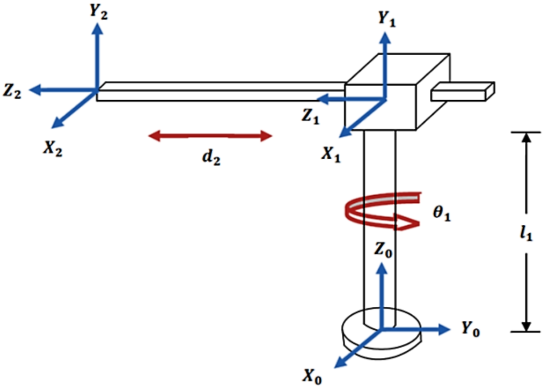
Inverse Kinematics:

$$\theta_1 = \tan^{-1}\frac{-x}{y}$$

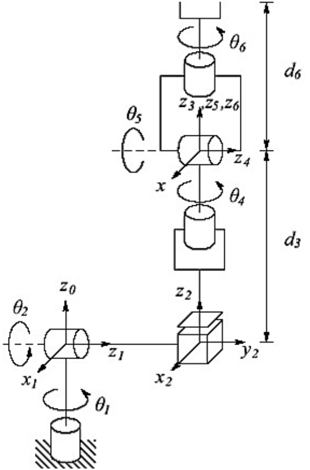
$$d_2 = \sqrt{x^2 + y^2}$$



Algebraic Approach:

Example: Stanford Manipulator

$$T_6^0 = A_1 \cdots A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Algebraic Approach:

Example: Stanford Manipulator

$$
\begin{aligned}
T_6^0 &= A_1 \cdots A_6 \\
&= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Not easy to find the IK in direct form!

$$
\begin{aligned}
r_{11} &= c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - d_2(s_4c_5c_6 + c_4s_6) \\
r_{21} &= s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) \\
r_{31} &= -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 \\
r_{12} &= c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) \\
r_{22} &= -s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) \\
r_{32} &= s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 \\
r_{13} &= c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 \\
r_{23} &= s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 \\
r_{33} &= -s_2c_4s_5 + c_2c_5 \\
d_x &= c_1s_2d_3 - s_1d_2 + +d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) \\
d_y &= s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) \\
d_z &= c_2d_3 + d_6(c_2c_5 - c_4s_2s_5)
\end{aligned}
$$

Questions?

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